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Quantum Central Limit Theorem for Continuous-Time Quantum Walks on Odd Graphs in Quantum Probability Theory

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Abstract The method of the quantum probability theory only requires simple structural data of graph and allows us to avoid a heavy combinational argument often necessary to obtain full description of spectrum of the adjacency matrix. In the present paper, by using the idea of calculation of the probability amplitudes for continuous-time quantum walk in terms of the quantum probability theory, we investigate quantum central limit theorem for continuous-time quantum walks on odd graphs.

Keywords Continuous-time quantum walk · Spectral distribution · Odd graph

1 Introduction

Two types of quantum walks, discrete and continuous time, were introduced as the quantum mechanical extension of the corresponding random walks and have been extensively studied over the last few years [1, 2].

Random walks on graphs are the basis of a number of classical algorithms. Examples include 2-SAT (satisfiability for certain types of Boolean formulas), graph connectivity, and finding satisfying assignments for Boolean formulas. It is this success of random walks that motivated the study of their quantum analogs in order to explore whether they might extend the set of quantum algorithms. Several systems have been proposed as candidates to implement quantum random walks. These proposals include atoms trapped in optical lattices [3], cavity quantum electrodynamics (CQED) [4] and nuclear magnetic resonance (NMR) in solid substrates [5, 6]. In liquid-state NMR systems [7, 8], time-resolved observations of spin waves has been done [9]. It has also been pointed out that a quantum walk can be simulated using classical waves instead of matter waves [10, 11].

A study of quantum walks on simple graph is well known in physics (see [12]). Recent studies of quantum walks on more general graphs were described in [13–19]. Some of these works studies the problem in the important context of algorithmic problems on graphs

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and suggests that quantum walks is a promising algorithmic technique for designing future quantum algorithms.

There is one approach for investigation of continuous-time quantum walk (CTQW) on graphs using the spectral distribution associated with the adjacency matrix of graphs [20–24]. Authors in [20] have introduced a new method for calculating the probability amplitudes of quantum walk based on spectral distribution which allows us to avoid a heavy combinational argument often necessary to obtain full description of spectrum of the Hamiltonian. It is interesting to investigate the CTQW on graph when the graph grows as time goes by. In order to study a quantum system in full detail, its Hamiltonian needs to be diagonalized. With increasing dimension of the Hilbert space, the diagonalization of an operator becomes a very tedious task. In fact, we discuss this question in the CTQW as a quantum central limit theorem for CTQW. In this paper we try to investigate quantum central limit theorem for CTQW on growing odd graph via spectral distribution.

The organization of the paper is as follows. In Sect. 2, we give a brief outline of graphs and introduce odd graph. In Sect. 3, we review the stratification and quantum decomposition for adjacency matrix of graphs. Section 4 is devoted to the method of computing the amplitude for CTQW, through spectral distribution μ of the adjacency matrix. In Sect. 4.1 we evaluate the CTQW on finite odd graph and in Sect. 4.2, we investigate quantum central limit theorem for CTQW on odd graph. The paper is ended with a brief conclusion and one Appendix which contains determination of spectral distribution by continued fractions method.

2 Odd Graph

In beginning we present a summery of graph and then we introduce odd graph.

A graph is a pair $G = (V, E)$, where V is a non-empty set and E is a subset of $\{\{\alpha, \beta\} | \alpha, \beta \in V, \alpha \neq \beta\}$. Elements of V and E are called *vertices* and *edges*, respectively. Two vertices $\alpha, \beta \in V$ are called adjacent if $\{\alpha, \beta\} \in E$, and in this case we write $\alpha \sim \beta$. Let $l^2(V)$ denotes the Hilbert space of C-valued square-summable functions on V, and $\{|\alpha\rangle | \ \alpha \in V\}$ becomes a complete orthonormal basis of $l^2(V)$. The adjacency matrix $A = (A_{\alpha\beta})_{\alpha,\beta \in V}$ is defined by

$$A_{\alpha\beta} = \begin{cases} 1 & \text{if } \alpha \sim \beta \\ 0 & \text{otherwise} \end{cases}$$

which is considered as an operator acting on $l^2(V)$ in such a way that

$$A|\alpha\rangle = \sum_{\alpha \sim \beta} |\beta\rangle, \quad \alpha \in V.$$

Obviously, (i) A is symmetric; (ii) an element of A takes a value in $\{0, 1\}$; (iii) a diagonal element of A vanishes. Conversely, for a non-empty set V, a graph structure is uniquely determined by such a matrix indexed by V. The *degree* or *valency* of a vertex $\alpha \in V$ is defined by

$$\kappa(\alpha) = |\{\beta \in V | \alpha \sim \beta\}|,$$

where $|.|$ denotes the cardinality.

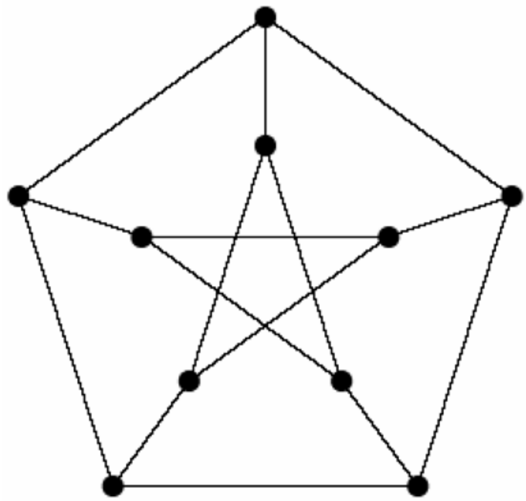

Fig. 1 The Petersen graph

Let S be a set of integer as $S = \{1, 2, \ldots, 2k - 1\}$ for a fixed integer $k \geq 2$. Now we define V be the set of subsets of S having cardinality $k - 1$, i.e.,

$$V = \{\alpha \subset S | \alpha| = k - 1\}, \tag{1}$$

and put

$$E = \{\{\alpha, \beta\} | \alpha, \beta \in V, \alpha \cap \beta = \emptyset\}. \tag{2}$$

This graph (V, E) is called the odd graph of degree k and is denoted by O_k. Obviously, O_k is a regular graph with degree k. For $k = 3$, the odd graph is well-known as Petersen graph (Fig. 1). The odd graphs have been studied in algebraic graph theory where some of their properties are found in [25, 26].

For $n = 0, 1, 2, \ldots, k - 1$ we define ϵ_n as

$$\epsilon_n = \begin{cases} k - 1 - \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{2} & \text{if } n \text{ is odd,} \end{cases}$$

then by using Proposition 4.1 [27], for pair $\alpha, \beta \in V$ we have

$$|\alpha \cap \beta| = \epsilon_n \Longleftrightarrow \partial(\alpha, \beta) = n, \tag{3}$$

where ∂ stands for the natural distance function. Due to the above relations based on distance function the odd graphs are distance regular graphs, i.e., for a given $i, j, l = 0, 1, 2, \ldots,$ the intersection number

$$p^l_{ij} = |\{\gamma \in V | \partial(\alpha, \gamma) = i \text{ and } \partial(\gamma, \beta) = j\}|, \tag{4}$$

is independently determined of the choice of $\alpha, \beta \in V$ satisfying $\partial(\alpha, \beta) = l$. There are some well-known facts about the intersection numbers of distance regular graphs, for example $p^i_{j1} = 0$ (for $i \neq 0$, j is not $\{i - 1, i, i + 1\}$) (for more details see [28, 29]). For convenience, set $b_i := p^i_{i-1,1}$ $(1 \leq i \leq d)$, $c_i := p^i_{i+1,1}$ $(0 \leq i \leq d - 1)$, $a_i := p^i_{i,1}$ $(0 \leq i \leq d)$,

$k_i := p^0_{i,i}$ $(0 \le i \le d)$ and $b_0 = c_d = 0$, where $d := \max\{\partial(\alpha, \beta) : \alpha, \beta \in V\}$ is called diameter of graph. Moreover

$$b_i + a_i + c_i = k, \quad 0 \le i \le d, \tag{5}$$

where $k = k_1$ is degree of graph.

For calculation of Szegö-Jacobi sequences $\{\omega_k\}$ and $\{\alpha_k\}$ in the next section we will require b_i and c_i [21], where for odd graph is given by

$$b_i = \begin{cases} \frac{i}{2} & \text{if } i \text{ is even,} \\ \frac{i+1}{2} & \text{if } i \text{ is odd} \end{cases} \quad (1 \le i \le k-1), \tag{6}$$

$$c_i = \begin{cases} k - \frac{i}{2} & \text{if } i \text{ is even,} \\ k - \frac{i+1}{2} & \text{if } i \text{ is odd} \end{cases} \quad (0 \le i \le k-2). \tag{7}$$

3 Quantum Probabilistic Approach for CTQW

In this section we give some preliminaries that require to describe CTQW via spectral distribution technique.

3.1 Stratification

Due to definition of function ∂, the graph becomes a metric space with the distance ∂. We fix a point $o \in V$ as an origin of the graph, called reference vertex. Then, the graph is stratified into a disjoint union of strata:

$$V = \bigcup_{i=0}^{\infty} V_i, \quad V_i = \{\alpha \in V \mid \partial(o, \alpha) = i\}. \tag{8}$$

With each stratum V_i we associate a unit vector in $l^2(V)$ defined by

$$|\phi_i\rangle = \frac{1}{\sqrt{|V_i|}} \sum_{\alpha \in V_i} |i, \alpha\rangle, \tag{9}$$

where $|i, \alpha\rangle$ denotes the eigenket of the αth vertex at the stratum i. The closed subspace of $l^2(V)$ spanned by $\{|\phi_i\rangle\}$ is denoted by $\Lambda(G)$. Since $\{|\phi_i\rangle\}$ becomes a complete orthonormal basis of $\Lambda(G)$, we often write

$$\Lambda(G) = \sum_i \oplus \mathbf{C}|\phi_i\rangle. \tag{10}$$

3.2 Quantum Decomposition

One can obtain a quantum decomposition associated with the stratification (8) for the adjacency matrices of this type of graphs as

$$A = A^+ + A^- + A^0, \tag{11}$$

where three matrices A^+, A^- and A^0 are defined as follows: for $\alpha \in V_i$

$$(A^+)_{\beta\alpha} = \begin{cases} A_{\beta\alpha} & \text{if } \beta \in V_{i+1} \\ 0 & \text{otherwise,} \end{cases}$$

$$(A^-)_{\beta\alpha} = \begin{cases} A_{\beta\alpha} & \text{if } \beta \in V_{i-1} \\ 0 & \text{otherwise,} \end{cases}$$

$$(A^0)_{\beta\alpha} = \begin{cases} A_{\beta\alpha} & \text{if } \beta \in V_{i} \\ 0 & \text{otherwise,} \end{cases}$$

or, equivalently, for $|i,\alpha\rangle$,

$$A^+|i,\alpha\rangle = \sum_{\beta\in V_{i+1}} |i+1,\beta\rangle, \qquad A^-|i,\alpha\rangle = \sum_{\beta\in V_{i-1}} |i-1,\beta\rangle, \qquad A^0|i,\alpha\rangle = \sum_{\beta\in V_{i}} |i,\beta\rangle, \tag{12}$$

for $\{\alpha,\beta\}\in E$. Since $\alpha\in V_i$ and $\{\alpha,\beta\}\in E$ then $\beta\in V_{i-1}\bigcup V_i\bigcup V_{i+1}$, where we tacitly understand that $V_{-1}=\emptyset$. The vector state corresponding to $|o\rangle=|\phi_0\rangle$, with $o\in V$ as the fixed origin, is analogous to the vacuum state in Fock space. According to [30], $\langle A^m\rangle$ coincides with the number of m-step walks starting and terminating at o, also, by Lemma 2.2, [30] if G is invariant under the quantum components A^ε, $\varepsilon\in\{+,-,0\}$, then there exist two Szegö-Jacobi sequences $\{\omega_i\}_{i=1}^{\infty}$ and $\{\alpha_i\}_{i=1}^{\infty}$ derived from A, such that

$$A^+|\phi_i\rangle = \sqrt{\omega_{i+1}}|\phi_{i+1}\rangle, \quad i\ge 0 \tag{13}$$

$$A^-|\phi_0\rangle = 0, \qquad A^-|\phi_i\rangle = \sqrt{\omega_i}|\phi_{i-1}\rangle, \quad i\ge 1 \tag{14}$$

$$A^0|\phi_i\rangle = \alpha_{i+1}|\phi_i\rangle, \quad i\ge 0, \tag{15}$$

where $\sqrt{\omega_{i+1}} = \frac{|V_{i+1}|^{1/2}}{|V_i|^{1/2}}\kappa_{-(\beta)}$, $\kappa_{-(\beta)} = |\{\alpha\in V_i|\alpha\sim\beta\}|$ for $\beta\in V_{i+1}$ and $\alpha_{i+1}=\kappa_{0(\beta)}$, such that $\kappa_{0(\beta)} = |\{\alpha\in V_i|\alpha\sim\beta\}|$ for $\beta\in V_i$. In particular $(\Lambda(G), A^+, A^-)$ is an interacting Fock space associated with a Szegö-Jacobi sequence $\{\omega_i\}$.

3.3 Study of CTQW on a Graph via Spectral Distribution of its Adjacency Matrix

The CTQW on graph has been introduced as the quantum mechanical analogue of its classical counterpart, which is defined by replacing Kolmogorov's equation (master equation) of continuous-time classical random walk on a graph [31, 32]

$$\frac{dP_m(t)}{dt} = \sum_{n=1}^{l} H_{mn}P_j(t), \quad m=1,2,\ldots,l \tag{16}$$

with Schrödinger's equation. Matrix H is the Hamiltonian of the walk and $P_m(t)$ is the occupying probability of vertex m at time t. It is natural to choose the Laplacian of the graph, defined as $L=A-D$ as the Hamiltonian of the walk, where D is a diagonal matrix with entries $D_{jj}=\deg(\alpha_j)$.

Let $|\phi(t)\rangle$ be a time-dependent amplitude of the quantum process on graph Γ. The wave evolution of the quantum walk is

$$i\hbar\frac{d}{dt}|\phi(t)\rangle = H|\phi(t)\rangle, \tag{17}$$

where from now on we assume $\hbar=1$, and $|\phi_0\rangle$ is the initial amplitude wave function of the particle. The solution is given by $|\phi(t)\rangle = e^{-iHt}|\phi_0\rangle$. On d-regular graphs, $D=\frac{1}{d}I$, and since A and D commute, we get

$$e^{-itH} = e^{-it(A-\frac{1}{d}I)} = e^{-it/d}e^{-itA}. \tag{18}$$

This introduces an irrelevant phase factor in the wave evolution. Hence we can consider $H = A$. Thus, we have

$$|\phi(t)\rangle = e^{-iAt}|\phi_0\rangle. \tag{19}$$

One of our goals in this paper is the evaluation of probability amplitudes for CTQW on graphs by using the method of spectral distribution associated with the adjacency matrix. The spectral properties of the adjacency matrix of a graph play an important role in many branches of mathematics and physics. The spectral distribution can be generalized in various ways. In this work, following [20, 30], we consider the spectral distribution μ of the adjacency matrix A:

$$\langle A^m\rangle = \int_R x^m \mu(dx), \quad m = 0, 1, 2, \ldots \tag{20}$$

where $\langle . \rangle$ is the mean value with respect to the state $|\phi_0\rangle$. By condition of quantum decomposition (QD) graphs the "moment" sequence $\{\langle A^m\rangle\}_{m=0}^{\infty}$ is well-defined [20, 30]. Then the existence of a spectral distribution satisfying (20) is a consequence of Hamburger's theorem, see e.g., Shohat and Tamarkin [34, Theorem 1.2].

Due to using the quantum decomposition relations (13–15) and the recursion relation (47) of polynomial $P_n(x)$, the other matrix elements $\langle\phi_n|A^m \mid \phi_0\rangle$ can be written as

$$\langle\phi_n|A^m \mid \phi_0\rangle = \frac{1}{\sqrt{\omega_1\omega_2\cdots\omega_n}} \int_R x^m P_n(x)\mu(dx), \quad m = 0, 1, 2, \ldots \tag{21}$$

where is useful for obtaining of amplitudes of CTQW in terms of spectral distribution associated with the adjacency matrix of graphs [20].

Therefore by using (21), the probability amplitude of observing the walk at stratum m at time t can be obtained as

$$q_m(t) = \langle\phi_m|\phi(t)\rangle = \langle\phi_m|e^{-iAt} \mid \phi_0\rangle = \frac{1}{\sqrt{\omega_1\omega_2\cdots\omega_m}} \int_R e^{-ixt} P_m(x)\mu(dx). \tag{22}$$

The conservation of probability $\sum_{m=0} \mid \langle\phi_m|\phi(t)\rangle \mid^2 = 1$ follows immediately from (22) by using the completeness relation of orthogonal polynomials $P_n(x)$. In the Appendix A of the reference [20], it is proved that the walker has the same amplitude at the vertices belonging to the same stratum, i.e., we have $q_{im}(t) = \frac{q_m(t)}{|V_m|}, i = 0, 1, \ldots, \mid V_m \mid$, where $q_{im}(t)$ denotes the amplitude of the walker at ith vertex of mth stratum.

Investigation of CTQW via spectral distribution method pave the way to calculate CTQW on infinite graphs and to approximate with finite graphs and vice versa, simply via Gauss quadrature formula, where in cases of infinite graphs, one can study asymptotic behavior of walk at large enough times by using the method of stationary phase approximation (for more details see [1]).

Indeed, the determination of $\mu(x)$ is the main problem in the spectral theory of operators, where this is quite possible by using the continued fractions method, as it is explained in Appendix.

4 Quantum Central Limit Theorem for CTQW on Odd Graphs

Having studied CTQW on finite odd graphs using the method of the spectral distribution, we investigate quantum central limit theorem for CTQW on this graphs which is our main goals.

To consider stratification and quantum decomposition of Sect. 3, [21] (i.e., $\omega_i = c_{i-1}b_i$, $\alpha_i = a_1 - b_{i-1} - c_{i-1}$) and (6), (7), we obtain two Szegö-Jacobi sequences $\{\omega_i\}$ and $\{\alpha_i\}$ for O_k as follows:
if i is odd,

$$\omega_i = \frac{i+1}{2}\left(k - \frac{i-1}{2}\right), \tag{23}$$

if i is even,

$$\omega_i = \frac{i}{2}\left(k - \frac{i}{2}\right) \tag{24}$$

if i is odd or even,

$$\alpha_i = 0. \tag{25}$$

4.1 Finite k Case

Let μ_k denote the spectral distribution of odd graph O_k. Here for studying CTQW on finite odd graph we consider $k = 4$ case. Then we have

$$\omega_1 = 4, \qquad \omega_2 = 3, \qquad \omega_3 = 6, \qquad \alpha_1 = \alpha_2 = \cdots = 0. \tag{26}$$

Therefore we obtain Stieltjes transform

$$G_{\mu_4}(z) = \frac{z^3 - 9z}{z^4 - 13z^2 + 24}. \tag{27}$$

In this case one can obtain the spectral distribution as follows

$$\begin{aligned}\mu_4 &= \frac{\sqrt{73}}{292}(5+\sqrt{73})\left(\delta\left(x - \frac{1}{2}\sqrt{26-2\sqrt{73}}\right) + \delta\left(x + \frac{1}{2}\sqrt{26-2\sqrt{73}}\right)\right)\\ &\quad + \frac{\sqrt{73}}{292}(-5+\sqrt{73})\left(\delta\left(x - \frac{1}{2}\sqrt{26+2\sqrt{73}}\right) + \delta\left(x + \frac{1}{2}\sqrt{26+2\sqrt{73}}\right)\right).\end{aligned} \tag{28}$$

By using (22) the amplitudes for walk at time t are

$$\begin{aligned}
q_o(t) &= \frac{\sqrt{73}}{146}\left((5+\sqrt{73})\cos\left(\frac{1}{2}\sqrt{26-2\sqrt{73}}\right)t + (-5+\sqrt{73})\cos\left(\frac{1}{2}\sqrt{26+2\sqrt{73}}\right)t\right),\\
q_1(t) &= \frac{-i\sqrt{73}}{584}\left((5+\sqrt{73})\left(\sqrt{26-2\sqrt{73}}\right)\sin\left(\frac{1}{2}\sqrt{26-2\sqrt{73}}\right)t\right.\\
&\quad \left. + (-5+\sqrt{73})\left(\sqrt{26+2\sqrt{73}}\right)\sin\left(\frac{1}{2}\sqrt{26+2\sqrt{73}}\right)t\right),\\
q_2(t) &= \frac{2\sqrt{3}}{\sqrt{73}}\left(-\cos\left(\frac{1}{2}\sqrt{26-2\sqrt{73}}\right)t + \cos\left(\frac{1}{2}\sqrt{26+2\sqrt{73}}\right)t\right),\\
q_3(t) &= \frac{-i\sqrt{73}}{584\sqrt{2}}\left(-(13+\sqrt{73})\left(\sqrt{26-2\sqrt{73}}\right)\sin\left(\frac{1}{2}\sqrt{26-2\sqrt{73}}\right)t\right.\\
&\quad \left. + (-13+\sqrt{73})\left(\sqrt{26+2\sqrt{73}}\right)\sin\left(\frac{1}{2}\sqrt{26+2\sqrt{73}}\right)t\right).
\end{aligned} \tag{29}$$

4.2 Quantum Central Limit Theorem

In the limit of large $k \longrightarrow \infty$, it is observed the odd graphs O_k form a growing family of distance regular graphs. In the remaining of this section, we obtain CTQW on odd graphs O_k for $k \longrightarrow \infty$ by applying the quantum central limit theorem where is derived from the quantum probabilistic techniques.

Theorem [27] *Let $\{G^k = (V^k, E^k)\}$ be a growing family of distance regular graphs. Let A_k and $\{p_{ij}^l(k)\}$ be the adjacency matrix and the intersection numbers of G^k, respectively. Assume that the limits*

$$\begin{aligned} \omega_i &= \lim_{k\longrightarrow\infty} \frac{p_{i-1,1}^i(k)\,p_{i,1}^{i-1}(k)}{p_{11}^0(k)} = \lim_{k\longrightarrow\infty} \frac{b_i(k)c_{i-1}(k)}{k} \\ \alpha_i &= \lim_{k\longrightarrow\infty} \frac{p_{i-1,1}^{i-1}(k)}{\sqrt{p_{11}^0(k)}} = \lim_{k\longrightarrow\infty} \frac{a_1(k)-b_{i-1}(k)-c_{i-1}(k)}{\sqrt{k}} \end{aligned} \tag{30}$$

exist for all $i = 1, 2, \ldots$. Let $\Lambda = (\Lambda, \{|\psi_i\rangle\}, B^+, B^-)$ be the interacting Fock space associated with $\{\omega_i\}$ and define a diagonal operator B^0 by $B^0|\psi_i\rangle = \alpha_{i+1}|\psi_i\rangle$. Therefor the quantum components $A_k^\varepsilon, \varepsilon \in \{+,-,0\}$, of adjacency matrix A_k is holds that

$$\lim_{k\longrightarrow\infty} \frac{A_k^\varepsilon}{\sqrt{p_{11}^0(k)}} = \lim_{k\longrightarrow\infty} \frac{A_k^\varepsilon}{\sqrt{k}} = B^\varepsilon \quad \varepsilon \in \{+,-,0\}, \tag{31}$$

in the stochastic sense. Then we have

$$\lim_{k\longrightarrow\infty} \langle\phi_m|\frac{A_k^\varepsilon}{\sqrt{k}}|\phi_0\rangle = \langle\psi_m|B^\varepsilon|\psi_0\rangle, \tag{32}$$

for $\varepsilon \in \{+,-,0\}$.

To state a quantum central limit theorem for CTQW on odd graph O_k, it is convenient to calculate amplitudes of probability as

$$q_m(t) = \lim_{k\longrightarrow\infty} \langle\phi_m|e^{\frac{-itA_k}{\sqrt{k}}}|\phi_0\rangle = \frac{1}{\sqrt{\omega_1\omega_2\ldots\omega_m}} \int_R e^{-ixt} P_m(x)\mu_\infty(dx). \tag{33}$$

According to the above theorem, we need only to find two Szegö-Jacobi sequences $\{\omega_i\}$ and $\{\alpha_i\}$, where by using (30) and (23) we obtain as follows:
if i is odd,

$$\omega_i = \lim_{k\longrightarrow\infty} \frac{1}{k}\frac{i+1}{2}\left(k - \frac{i-1}{2}\right) = \frac{i+1}{2}, \tag{34}$$

if i is even,

$$\omega_i = \lim_{k\longrightarrow\infty} \frac{1}{k}\frac{i}{2}\left(k - \frac{i}{2}\right) = \frac{i}{2}, \tag{35}$$

if i is odd or even,

$$\alpha_i = 0. \tag{36}$$

Thus, $\{\omega_i\} = \{1, 1, 2, 2, 3, 3, 4, 4, \ldots\}$ as desired. Therefore Stieltjes transform of infinite odd graphs is

$$G_{\mu_\infty}(z) = \cfrac{1}{z - \cfrac{1}{z - \cfrac{1}{z - \cfrac{2}{z - \cfrac{2}{z - \cfrac{3}{z - \cfrac{3}{z - \cdots}}}}}}}, \tag{37}$$

where the spectral distribution $\mu_\infty(x)$ in the Stieltjes transform is given by [27]

$$\mu_\infty(x) = |x| e^{-x^2}. \tag{38}$$

Therefore we obtain the amplitude of probability at time t and the 0-th stratum (starting vertex)

$$\begin{aligned} q_0(t) &= \int_R e^{-ixt} \mu_\infty(dx) = \int_{-\infty}^{\infty} e^{-ixt} |x| e^{-x^2} dx \\ &= \frac{i\sqrt{\pi} t}{2} \operatorname{erf}(it/2) e^{-\frac{t^2}{4}} + 1. \end{aligned} \tag{39}$$

In the above calculation we used formulas:

$$\begin{aligned} x^n e^{-ixt} &= i^n \frac{d^n}{dt^n} e^{-ixt}, \\ \int_0^{\infty} e^{-x^2} e^{-ixt} dx &= \frac{\sqrt{\pi}}{2} (1 - \operatorname{erf}(it/2)) e^{\frac{-t^2}{4}}, \end{aligned} \tag{40}$$

where $\operatorname{erf}(x)$ stands for the error function and it is defined as:

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-s^2} ds \tag{41}$$

also, the derivative of the error function follows immediately from its definition $\frac{\partial}{\partial x} \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} e^{-x^2}$.

By using (22) and (40), we obtain the amplitude of probability for walk at time t and m-th stratum on infinite odd graphs in terms of $q_0(t)$ as:
if m is odd

$$q_m(t) = \frac{1}{(\frac{m-1}{2})!(\frac{m+1}{2})!} P_m\left(i \frac{d}{dt}\right) q_0(t), \tag{42}$$

if m is even

$$q_m(t) = \frac{1}{((\frac{m}{2})!)^2} P_m\left(i \frac{d}{dt}\right) q_0(t), \tag{43}$$

where polynomials $\{P_m(i\frac{d}{dt})\}$ are defined recurrently by relation (47) in terms of $i\frac{d}{dt}$. In the end for example we obtain the amplitude of probability for walk at stratum 1, 2 and 3 as follows:
$m = 1$

$$q_1(t) = P_1\left(i \frac{d}{dt}\right) q_0(t) = i \frac{d}{dt} q_0(t) = \frac{\sqrt{\pi}}{4} (t^2 - 2) \operatorname{erf}(it/2) e^{\frac{-t^2}{4}} - it/2, \tag{44}$$

$m = 2$

$$q_2(t) = P_2\left(i\frac{d}{dt}\right)q_0(t) = \left(\left(i\frac{d}{dt}\right)^2 - 1\right)q_0(t)$$
$$= \frac{1}{8}\left(-i\sqrt{\pi}t^3\,\mathrm{erf}(it/2)e^{\frac{-t^2}{4}} - 2t^2 + 2i\sqrt{\pi}t\,\mathrm{erf}(it/2)e^{\frac{-t^2}{4}}\right), \tag{45}$$

$m = 3$

$$q_3(t) = \frac{1}{2}P_3\left(i\frac{d}{dt}\right)q_0(t) = \frac{1}{2}\left(\left(i\frac{d}{dt}\right)^3 - 2i\frac{d}{dt}\right)q_0(t)$$
$$= \frac{1}{16\cdot 2}\Big(-\sqrt{\pi}t^4\mathrm{erf}(it/2)e^{\frac{-t^2}{4}} + 2it^3 + 4\sqrt{\pi}t^2\mathrm{erf}(it/2)e^{\frac{-t^2}{4}}$$
$$-4it + 4\sqrt{\pi}\mathrm{erf}(it/2)e^{\frac{-t^2}{4}}\Big). \tag{46}$$

5 Conclusion

In this paper by using the method of calculation of the probability amplitude for continuous-time quantum walk on graph via quantum probability theory, we have studied continuous-time quantum walk on odd graph when the odd graph grow as time goes by. We have discussed this question as a quantum central limit theorem for CTQW. It is interesting to investigate CTQW on a growing family of graphs since it is probable the probability amplitudes of CTQW to converge the uniform distribution, which is under investigation.

Appendix

Determination of Spectral Distribution by Continued Fractions Method

In this appendix we explain how we can determine spectral distribution $\mu(x)$ of the graphs, by using the Szegö-Jacobi sequences $(\{\omega_k\}, \{\alpha_k\})$, which the parameters ω_k and α_k are defined in the Sect. 3.2.

To this aim we may apply the canonical isomorphism from the interacting Fock space onto the closed linear span of the orthogonal polynomials determined by the Szegö-Jacobi sequences $(\{\omega_i\}, \{\alpha_i\})$. More precisely, the spectral distribution μ under question is characterized by the property of orthogonalizing the polynomials $\{P_n\}$ defined recurrently by

$$P_0(x) = 1, \qquad P_1(x) = x - \alpha_1,$$
$$xP_n(x) = P_{n+1}(x) + \alpha_{n+1}P_n(x) + \omega_n P_{n-1}(x), \tag{47}$$

for $n \geq 1$.

As it is shown in [33], the spectral distribution can be determined by the following identity:

$$G_\mu(z) = \int_R \frac{\mu(dx)}{z-x} = \frac{1}{z-\alpha_1-\frac{\omega_1}{z-\alpha_2-\frac{\omega_2}{z-\alpha_3-\frac{\omega_3}{z-\alpha_4-\cdots}}}} = \frac{Q_{n-1}^{(1)}(z)}{P_n(z)} = \sum_{l=1}^{n}\frac{A_l}{z-x_l}, \tag{48}$$

where $G_\mu(z)$ is called the Stieltjes transform and A_l is the coefficient in the Gauss quadrature formula corresponding to the roots x_l of polynomial $P_n(x)$ and where the polynomials $\{Q_n^{(1)}\}$ are defined recurrently as

$$Q_0^{(1)}(x) = 1,$$
$$Q_1^{(1)}(x) = x - \alpha_2,$$
$$xQ_n^{(1)}(x) = Q_{n+1}^{(1)}(x) + \alpha_{n+2}Q_n^{(1)}(x) + \omega_{n+1}Q_{n-1}^{(1)}(x),$$

for $n \geq 1$.

Now if $G_\mu(z)$ is known, then the spectral distribution μ can be recovered from $G_\mu(z)$ by means of the Stieltjes inversion formula:

$$\mu(y) - \mu(x) = -\frac{1}{\pi} \lim_{v \longrightarrow 0^+} \int_x^y \operatorname{Im}\{G_\mu(u + iv)\}du. \tag{49}$$

Substituting the right hand side of (48) in (49), the spectral distribution can be determined in terms of $x_l, l = 1, 2, \ldots,$ the roots of the polynomial $P_n(x)$, and Gauss quadrature constant $A_l, l = 1, 2, \ldots$ as

$$\mu = \sum_l A_l \delta(x - x_l) \tag{50}$$

(for more details see [20, 21, 33, 34]).

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